An RMT Approach to a Free Strong Szegő Limit Theorem

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Random Matrices: Expectations

- A central topic in Random Matrix Theory is the study of eigenvalues of large dimensional random matrices initiated by work of Wishart (1928, [26]) and Wigner (1955, [25]).
- Here is an illustration of Wigner's result:

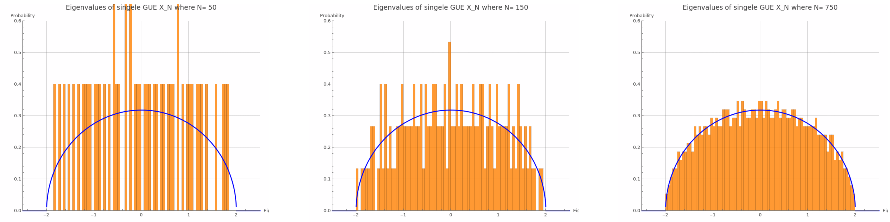

Figure: Eigenvalues of a $N \times N$ selfadjoint Gaussian random matrix at size $N = 50$, 150, and 750.

Random Matrices: Expectations

Let X_N be an $N \times N$ selfadjoint random matrix whose upper-triangular entries are i.i.d. standard complex Gaussian and scaled by $\frac{1}{\sqrt{N}}$.

Theorem (Wigner 1955, [25])

For any integer $k \geq 1$,

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} Tr\left(X_N^k\right)\right] = \frac{1}{2\pi}\int_{-2}^{2} x^k \sqrt{4-x^2}dx.$$

Whenever the above conditions hold for $X = (X_N)_{N\in\mathbb{N}}$, we say that the random matrix ensemble X has a **limiting distribution**.

Random Matrices: Expectations

We say a unitary valued random matrix U_N is **Haar distributed** when its distribution is given by normalized Haar measure on the unitary group $\mathcal{U}(N)$.

Theorem

For any integer $k \neq 0$,

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} Tr\left(U_N^k\right)\right] = 0$$

which is exactly the moments of the uniform probability on the unit circle.

Random Matrices: Fluctuations

Now that we know the asymptotic eigenvalue distribution, we can ask about the fluctuations around these asymptotics. [1]

Theorem (Fluctuations for selfadjoint Gaussian r.m.'s)

For any polynomial p,

$$Tr\, p(X_N) - \mathbb{E}\left[p\left(X_N\right)\right] \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, \sigma_p^2\right)$$

[1] Started in 1982 by Jonsson [11], later in 1998 Johansson [9], and refined in 2001 by Cabanal-Duvillard [3] .

Random Matrices: Fluctuations

Theorem (Diaconis-Shashahani 1994, [7])

For any polynomial $p(1) = 0$,

$$Tr\ p(U_N) \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, \sigma_p^2\right)$$

Free Probability

- In 1983 Voiculescu [23] developed a non-commutative analog of probability theory know as free probability.
- His goal was to answer an operator algebraic question know as the "free group factor" problem.
- In 1991 [22] he discovered that classically independent random matrices become "freely independent" in the large dimensional limit.

Random Matrices | **Free Probability** | **Operator Algebras**

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Asymptotic Freeness

Theorem (Voiculescu 1991, [22])

- *Independent selfadjoint Gaussian random matrices* $(X_1^{(d)}, \ldots, X_g^{(d)})$ *are asymptotically free.*
- *Independent Haar-distributed random unitary matrices* $(U_1^{(d)}, \ldots, U_g^{(d)})$ *are asymptotically free.*

where we say

Definition (Asymptotic Freeness $g = 2$)

Two random matrix ensembles $X = (X_N)_{N\in\mathbb{N}}$ and $Y = (Y_N)_{N\in\mathbb{N}}$ with limit eigenvalue distributions are said to be **asymptotically free** if for all $p \geq 1$ and all $n_1, m_1, \ldots, n_p, m_p \geq 1$, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\operatorname{tr}\left(X_N^{n_1} - \alpha_{n_1}^X\right)\left(Y_N^{m_1} - \alpha_{m_1}^Y\right)\cdots\left(X_N^{n_p} - \alpha_{n_p}^X\right)\left(Y_N^{m_p} - \alpha_{m_p}^Y\right)\right] = 0.$$

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Free Probability

- In 1994, Speicher [19] developed a combinatorial framework for Voiculescu's work on freeness and limiting distributions using non-crossing partitions. In particular a combinatorial description for "free cumulants".
- From 2006-2013 the relationship random matrix theory and free probability at the level of fluctuations ([16],[15],[17],[14]) and leading to the notion of *second-order asymptotic freeness*[2].
- First-order free probability describes limiting moments, while second-order free probability captures fluctuation moments, that is, limits of covariances.

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Weak and Strong Convergence

1. It turns that a lot of the important random matrix models are almost surely asymptotically free.
2. For the Haar unitary ensemble this means for any polynomial p in g non-commuting indeterminates

$$\frac{1}{d}\operatorname{Tr} p\left(U_1^{(d)}, \cdots, U_g^{(d)}\right) \to \tau\left(p\left(u_1, \cdots, u_g\right)\right) \text{ almost surely}$$

where $u_1, \cdots, u_g$ "freely independent" operators living in some "non-commutative probability space" $(\mathcal{A}, \tau)$ (more later).[3]

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3. Can we replace the normalized trace with other functions on matrices? Such as norms?

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Weak and Strong Convergence

1. Yes, we have

$$\left\| p\left(U_1^{(d)}, \cdots, U_g^{(d)}\right) \right\| \to \|\tau\left(p\left(u_1, \cdots, u_g\right)\right)\| \text{ almost surely}$$

where $u_1, \cdots, u_g$ "freely independent" operators living in some "non-commutative probability space" $(\mathcal{A}, \tau)$ (more later).[3]

2. The breakthrough came in 2005 from Haagerup–Thorbjørnsen [8] which established the above result for independent selfadjoint Gaussian random matrices. Later, in 2014, Collins and Male [5] extended this result to the Haar unitary setting.

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Why do we care?

- Random matrix theory and free probability provide a new framework to studying classical problems in complex analysis and operator theory, such as the strong Szegő limit theorem.
- The novelty lies in the fact that none of the single-variable techniques extend to a multivariate analog, necessitating a fundamentally new approach.
- This will be the focus for the remainder of the talk.

Toeplitz Matrices

Let $f : \mathbb{T} \to \mathbb{C}$ continuous, and for each integer n, denote its Fourier coefficients by $c_n := \hat{f}(n)$. The matrix

$$T_d(f) = \begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-(d-1)} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-(d-2)} \\ c_2 & c_1 & c_0 & \cdots & c_{-(d-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(d-1)} & c_{d-2} & c_{d-3} & \cdots & c_0 \end{bmatrix}$$

is called the **truncated Toeplitz matrix** associated with the symbol f.

The Classical Strong Szegő Limit Theorem

The **Strong Szegő Limit Theorem**[4] is a statement about the asymptotic behavior of Toeplitz determinants as the matrix size $d \to \infty$.

Theorem (SSLT in Determinental Form)

Let $f : \mathbb{T} \to \mathbb{C}$ be continuous. Suppose the Fourier coefficients c_n satisfy:

$$\sum_{n \in \mathbb{Z}} |c_n| < \infty \quad \text{and} \quad \sum_{n \in \mathbb{Z}} |n||c_n|^2 < \infty$$

Then, as $d \to \infty$:

$$\log \det T_d(e^f) = dc_0 + \frac{1}{2} \sum_{n \in \mathbb{Z}} |n| c_n c_{-n} + o(1)$$

[4] The first proof was given by Gábor Szegő in 1952 [21].

A Probabilistic Approach to Strong Szegő

Remarkably, the same coefficients appearing in the Strong Szegő Theorem also govern the fluctuations of linear eigenvalue statistics for random unitary matrices.

A Probabilistic Approach to Strong Szegő

"Coulomb gas representation" (1915 [20])

Let $f \in L^1(\mathbb{T})$. Then

$$\det T_d(f) = \frac{1}{d!(2\pi)^d} \int_{\mathbb{T}^d} \prod_{\mu=1}^{d} f(\theta_\mu)\, \Delta(\theta)^2\, \mathrm{d}\theta,$$

Weyl integration formula (1925 [24])

Let $F : \mathcal{U}(d) \to \mathbb{C}$ be an integrable class function. Then

$$\mathbb{E}[F(U)] = \frac{1}{d!(2\pi)^d} \int_{\mathbb{T}^d} F(\theta_1, \cdots, \theta_d)\, \Delta(\theta)^2\, \mathrm{d}\theta,$$

where

$$\Delta(\theta) := \prod_{1 \le \mu < \nu \le d} |e^{i\theta_\mu} - e^{i\theta_\nu}|.$$

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A Probabilistic Approach to Strong Szegő

- In particular for continuous symbol $f : \mathbb{T} \to \mathbb{C}$,

$$\det T_d\left(e^f\right) = \mathbb{E}_d\left[\det\left(\exp f\left(U\right)\right)\right].$$

- The probabilistic approach to the Strong Szegő Limit Theorem relies on averaged determinants over the unitary group $\mathcal{U}(d)$, with no mention of Toeplitz matrices.

A Probabilistic Approach to Strong Szegő

Probabilistic approaches to SSLT were developed by:

- **Johansson (1988 [10]):** Introduced the probabilistic framework.
- **Bump–Diaconis (2002 [2]):** A combinatorial proof using symmetric functions.

Theorem (SSLT in Probabilistic Form)

Let $f : \mathbb{T} \to \mathbb{C}$ be continuous. Suppose the Fourier coefficients c_n satisfy:

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Then, as $d \to \infty$:

$$\log \mathbb{E}_d \left[\exp\left(\operatorname{Tr} f(U)\right)\right] = dc_0 + \frac{1}{2}\sum_{n\in\mathbb{Z}} |n| c_n c_{-n} + o(1) \tag{1}$$

A Fluctuation Results

Corollary (Fluctuations of tracial power series)

Suppose the sequence (c_n) *satisfy* $\sum_{n\in\mathbb{Z}} |c_n|,\ \sum_{n\in\mathbb{Z}} |n||c_n|^2 < \infty$. *Let*

$$G_d := Tr \sum_{n\in\mathbb{Z}} c_n U^{(d)n}.$$

Assume in addition that $c_n = \overline{c_{n^{-1}}}$ *(so that* G_d *is real-valued). Then for all real numbers* t *we have*

$$\lim_{d\to\infty} \left(\log \mathbb{E}\left[\exp(tG_d)\right] - t\mathbb{E}G_d\right) = \left(\frac{1}{2}\sum_{n\in\mathbb{Z}} |c_n|^2 |n|\right) t^2. \qquad (2)$$

In particular, the sequence $(G_d - \mathbb{E}G_d)_{d=1}^{\infty}$ *converges in distribution to a normal random variable with mean* 0 *and variance* $\frac{1}{2}\sum_{n\in\mathbb{Z}} |c_n|^2 |n|$.

NC-Functions

- Let $\mathbb{F}_g$ denote the **free group** on g generators $\{\gamma_1, \cdots, \gamma_g\}$.
- Given a reduced word $w \in \mathbb{F}_g$, let U^w denote the corresponding monomial in the U_i and U_i^*.
- Example: if $w = \gamma_2\gamma_1^{-1}\gamma_2\gamma_3$, then

$$U^w = U_2 U_1^* U_2 U_3.$$

- We denote the **length** of the reduced word by $|w|$.

An "Easy" Free Strong Szegő Limit Theorem

Theorem (Jury, Roman, v.R.)

Suppose $c : \mathbb{F}_g \to \mathbb{C}$ *is admissible*[a], *and* $c_\varnothing = 0$. *If*

$$\sum_{w \in \mathbb{F}_g} |c_w|\,|w| < +\infty$$

then

$$\lim_{d\to\infty} \mathbb{E}_d \left[\exp\left(Tr \sum_{w\in\mathbb{F}_g} c_w U^w \right)\right] = \exp\left(\frac{1}{2} \sum_{w\in\mathbb{F}_g} c_w c_{w^{-1}} |w| \right). \qquad (3)$$

[a] A technical assumption not needed for existence of the limit; imposed to obtain a simpler expression for the limit.

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In particular, the sequence $(G_d - \mathbb{E}G_d)_{d=1}^{\infty}$ converges in distribution to a normal random variable with mean 0 and variance $\frac{1}{2}\sum_{w \in \mathbb{F}_g} |c_w|^2 |w|$.

Proof Outline

Step 1: Second order asymptotic freeness

- First suppose c is finitely supported.
- Second order asymptotic freeness [15] gives you

$$\exp\left(\mathrm{Tr}\sum_{w\in\mathbb{F}_g} c_w U^{(d)w}\right) \xrightarrow{\text{dist}} \exp(Z)$$

  where Z is a centered, complex Gaussian.

- Moreover we can compute the pseudo-variance

$$\mathbb{E}\left[Z^2\right] = \sum_{w\in\mathbb{F}_g} c_w c_{w^{-1}} |w|$$

  by means of "second order freeness".

Proof Outline

Step 2: Concentration

Theorem (Meckes–Meckes; 2013 [12])

Let $F : \mathcal{U}(d)^g \to \mathbb{R}$ be a L-Lipschitz function denote $U_1, \ldots, U_g$ independent Haar-distributed $d \times d$ unitary random matrices. Then

$$\mathbb{P}\left(|F(U_1, \ldots, U_g)| \geq t \right) \leq 2\exp\left(-\frac{d\left(t - |\mathbb{E}F(U_1, \ldots, U_g)|\right)^2}{24L^2} \right).$$

for every $t > |\mathbb{E}F(U_1, \ldots, U_g)|$.

□

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1. By an concentration argument, the sequence $\mathrm{Tr} \sum_{w \in \mathbb{F}_g} c_w U^{(d)w}$ has sub-gaussian tail bounds uniform in d, and we obtain convergence of means as in the statement.

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And the general case follows from an approximation argument again using the concentration phenomena. □

Linear Pencils

- Next we ask for which nc-functions does our "easy" SSLT apply?
- We define the evaluation of the monic linear pencil L_A on a g-tuple of $d \times d$ matrices X as follows:

$$L_A(X) := I_k \otimes I_d + \sum_{i=1}^{g} A_i \otimes X_i.$$

Here, the coefficients A_i are $k \times k$ matrices, and the $X_1, \cdots, X_g$ are $d \times d$ matrices, and $\otimes$ denotes the Kronecker product.

Consequences of "easy" SSLT

- **Classical:** If $\rho(A) < 1$, then classical Strong Szegő imply

$$\lim_{d\to\infty} \mathbb{E}_d \left[\left| \det \left(I_k \otimes I_d + A \otimes U \right) \right|^2 \right] = \det \left(I - A \otimes \overline{A} \right)^{-1}$$

where A a $k \times k$ matrix.

- **Free:** If $\sup_U \|L_A(U)\| < 1$, then "easy" Strong Szegő imply

$$\lim_{d\to\infty} \mathbb{E}_d \left[\left| \det \left(I_k \otimes I_d + \sum_{i=1}^{g} A_i \otimes U_i \right) \right|^2 \right] = \det \left(I - \sum_{i=i}^{g} A_i \otimes \overline{A_i} \right)^{-1}$$

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Domain of Convergence

- If $\sup_U \|L_A(U)\| < 1$, define c for $w \in \mathbb{F}_g^+$ by

$$c_w = \frac{(-1)^{|w|+1}\mathrm{Tr}(A^w)}{|w|}, \quad \text{and} \quad c_{w^{-1}} = \frac{(-1)^{|w|+1}\overline{\mathrm{Tr}(A^w)}}{|w|}$$

and apply "easy" SSLT.

- Although

$$\sup_U \|L_A(U)\| < 1 \implies \rho\left(\sum_{i=1}^{g} A_i \otimes \overline{A_i}\right)^{\frac{1}{2}} < 1$$

the converse is not true in general.

A Multivariate Phenomenon (g=1 and g=2)

Eigenvalues of $L_A(U)$ with $\rho(A) < 1$ and $\rho(\sum_{i=1}^{2} A_i \otimes \overline{A_i})^{1/2} < 1$:

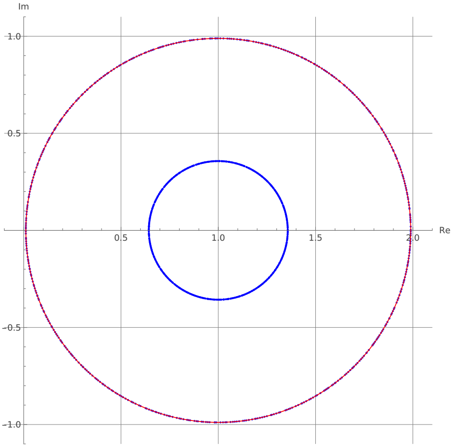
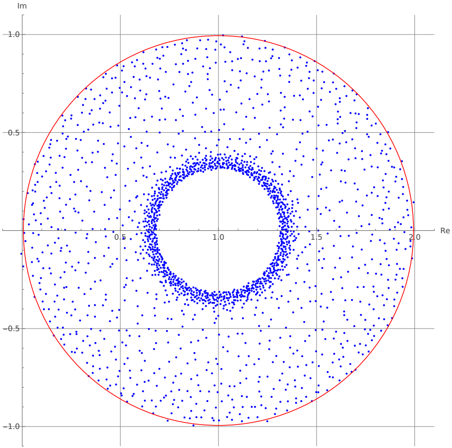


Figure: Single variable



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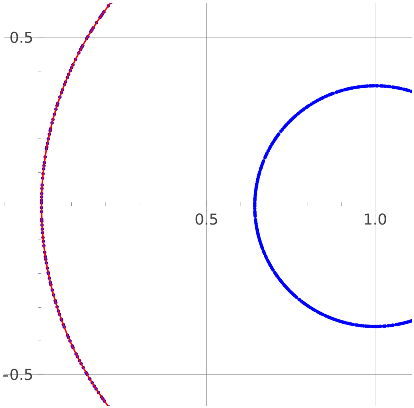

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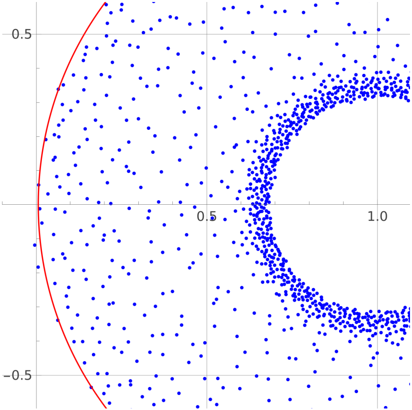

Figure: Multivariate

A Free Strong Szegő Limit Theorem

Theorem (Jury, Roman, v.R.)

Given g-tuples square matrices of $(A_1, \cdots, A_g)$, and $(B_1, \cdots, B_g)$ of size $k \times k$, and $\ell \times \ell$ respectively. If

$$\rho\left(\sum_{i=1}^{g} A_i \otimes \overline{A_i}\right) < 1, \quad \rho\left(\sum_{i=1}^{g} B_i \otimes \overline{B_i}\right) < 1,$$

then

$$\lim_{d\to\infty} \mathbb{E}_d\left[\det L_A(U)\, L_B(U)^*\right] = \det\left(I_k \otimes I_\ell - \sum_{i=1}^{g} A_i \otimes \overline{B_i}\right)^{-1}.$$

Proof Outline

Main Tools

1. Second-order asymptotic freeness
2. Concentration of Measure
3. Strong Asymptotic Expansions (Strong Convergence)
4. Spectral Information of the Limiting Operator

Spectral Information of the Limiting Operator

- Denote by

$$\lambda : \mathbb{F}_g \to \mathcal{U}(\ell^2(\mathbb{F}_g))$$

the **left regular representation** of the free group.

- The **reduced C^*-algebra**

$$C^*_\lambda(\mathbb{F}_g)$$

is the C^*-algebra generated by $\lambda(\mathbb{F}_g)$.

- Denote

$$u_i := \lambda(\gamma_i)$$

which we will refer to as **free Haar unitaries**.

Spectral Information of the Limiting Operator

Free probability says that $\sum_{i=1}^{g} A_i \otimes u_i$ is the correct limiting operator for the random matrix $\sum_{i=1}^{g} A_i \otimes U_i^{(d)}$.

Theorem (Jury, v.R. - 2025)

Let $A_1, \ldots, A_g \in M_k(\mathbb{C})$. Then the spectral radius of the operator $\sum_{i=1}^{g} A_i \otimes u_i$ is given by $\rho\left(\sum_i \overline{A_i} \otimes A_i \right)^{\frac{1}{2}}$.

Further Questions 1: Other Ensembles

Recall the main tools:

1. Second-order asymptotic freeness
2. Concentration of Measure
3. Strong Asymptotic Expansions (Strong Convergence)
4. Spectral Information of the Limiting Operator

Further Questions 1: Other Ensembles

Haar orthogonal random matrices

1. Second-order asymptotic freeness
   - Haar orthogonal random matrices are real-second order asymptotically free [14].
2. Concentration of Measure
   - Haar orthogonal random matrices satisfy a concentration phenomena [13].
3. Strong Asymptotic Expansions (Strong Convergence)
   - Haar orthogonal random matrices satisfy the strong convergence phenomena [4].
4. Spectral Information of the Limiting Operator
   - In progress.

Further Questions 1: Other Ensembles

Non-selfadjoint Gaussian random matrices

1. Second-order asymptotic freeness
   - Follows from the selfadjoint case [16], and second order limit are called "circular operators".
2. Concentration of Measure
   - Follows from Gaussian concentration [1].
3. Strong Asymptotic Expansions (Strong Convergence)
4. Spectral Information of the Limiting Operator
   - The spectral radius computation hold for "circular operators" [6].

Further Questions 2: Higher order Szegő Asymptotics

We have shown

$$\log \mathbb{E}_d \left[\det L_A(U) L_A(U)^*\right] = d \log\det(A_0 A_0^*) + \log\det\left(I_k \otimes I_k - \sum_{i=1}^{g} \tilde{A}_i \otimes \overline{\tilde{A}_i}\right)^{-1} + o(1)$$

- One may ask whether there exists a systematic expansion in $\frac{1}{d}$, and if so, what the next coefficients in the series would be.
- In the single-variable case [18] the error term is $O(e^{-Bd})$, thus no further coefficients.
- In the multivariate setting the rate of unitary fluctuations is significantly slower, of order $O\left(\frac{1}{d}\right)$ compared to the single-variable case where the asymptotics stabilize in finitely many steps.

Thank You For Listening!

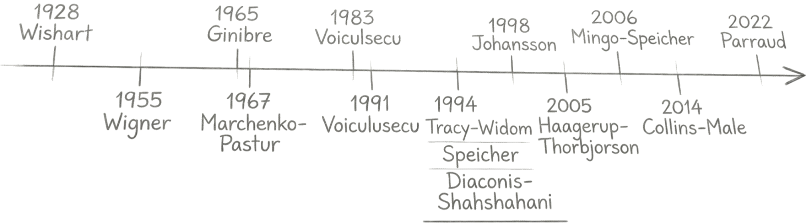

Figure: Historical Progress of RMT and Free Probability

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