The Spectral Radius of Free Haar Unitary Pencils

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Joint work with Mike Jury and George Roman

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Introduction

- Fix an integer $d \geq 1$.
- Let $\mathbb{F}_d$ denote the **free group** on d generators $\{g_1, g_2, \ldots, g_d\}$.
- Denote by
$$\lambda : \mathbb{F}_d \to \mathcal{U}(\ell^2(\mathbb{F}_d))$$
the **left regular representation** of the free group.
- The **reduced** C^***-algebra**
$$C^*_\lambda(\mathbb{F}_d)$$
is the C^*-algebra generated by $\lambda(\mathbb{F}_d)$.

Introduction

- Denote
$$u_i := \lambda(g_i)$$
which we will refer to as **free Haar unitaries**.
- Fix $X_1, \ldots, X_d$ bounded operators on some Hilbert space $\mathcal{H}$. By a **linear pencil** in free Haar unitaries we mean
$$X_1 \otimes u_1 + X_2 \otimes u_2 + \ldots + X_d \otimes u_d$$
where $\otimes$ refers to the minimal tensor product.

Preliminaries

- Given a reduced word $w = g_{i_1}...g_{i_k}$ (not inverses) we denote

  $$X^w = X_{i_1}...X_{i_k}$$

  and

  $$\lambda(g)^w = \lambda(g_{i_1})...\lambda(g_{i_k}).$$
- For example if $d = 3$, and $w = g_2g_1g_3g_3$, then

  $$X^w = X_2X_1X_3X_3.$$
- We denote the **length** of the reduce word by $|w|$.

Motivation

- A central question in Random Matrix Theory is to study the eigenvalues of a random matrix.
- In a project with Mike Jury and George Roman, we are interested in the eigenvalues of random matrices of the form

  $$X_1 \otimes U_1^{(n)} + \cdots + X_d \otimes U_d^{(n)}$$

  as $n \to \infty$.
- Here, $U_i^{(n)}$ are independent, $n \times n$, Haar-distributed, unitary random matrices, and $X_i \in M_\ell(\mathbb{C})$ are fixed coefficients.

Motivation

- There is a sense in which the deterministic operator

  $$X_1 \otimes u_1 + X_2 \otimes u_2 + \cdots + X_d \otimes u_d$$

  is the limiting object of the random unitary pencil.
- This notion is made precise by the work of Voiculescu in 1991 [11], further developed by Collins and Male [3] in 2014, and Parraud in 2022 [7].
- Our approach to understanding the eigenvalues of the large random matrices is to study the spectrum of this deterministic operator. In particular compute its **spectral radius**.

Example

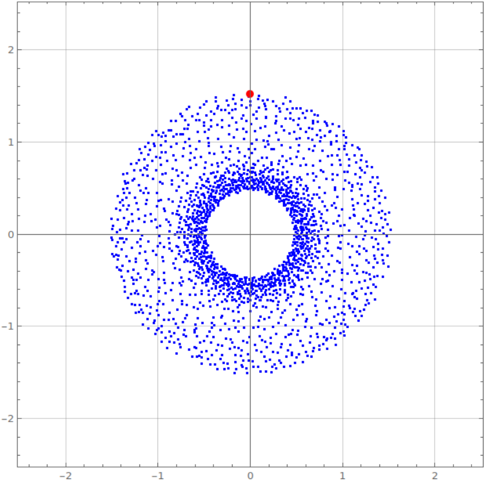

Figure: Eigenvalues of a single realization of $X_1 \otimes U_1^{(n)} + X_2 \otimes U_2^{(n)}$, where Haar unitary matrices with $n = 1000$, and $X_1, X_2 \in M_2(\mathbb{C})$.

Classical Haagerup's Inequality

- In 1979, Haagerup [5] proved

$$\| \sum_{g\in W_n} \alpha_g \lambda(g) \| \leq (n+1)(\sum_{g\in W_n} |\alpha_g|^2)^{\frac{1}{2}}$$

for any $(\alpha_g) : \mathbb{F}_g \to \mathbb{C}$, where $W_n \subset \mathbb{F}_d$ of all reduced words of length n.

- By $W_n^+ \subseteq \mathbb{F}_d^+$ we denote all words of length n in generators, and not inverses. Here $\mathbb{F}_d^+$ denotes the free monoid on d generators.

Generalized Haagerup's Inequality

- In 1993, Haagerup and Pisier [6] generalized to operator valued coefficients.
- They showed

$$\| \sum_g \alpha_g \otimes \lambda(g) \| \geq \max\{\| \sum_g \alpha_g^* \alpha_g \|^{\frac{1}{2}} , \| \sum_g \alpha_g \alpha_g^* \|^{\frac{1}{2}} \}$$

  for any finitely supported $(\alpha_g) : \mathbb{F}_d \to B(\mathcal{H})$.
- Also if (α_g) is supported on $\{g_1, .., g_d, g_1^{-1}, .., g_d^{-1}\}$, then

$$\| \sum_g \alpha_g \otimes \lambda(g) \| \leq 2 \max\{\| \sum_g \alpha_g^* \alpha_g \|^{\frac{1}{2}} , \| \sum_g \alpha_g \alpha_g^* \|^{\frac{1}{2}} \}$$

Generalizations of Haagerup's Inequality

- By Gelfand's Formula we need to estimate n^{th} roots of norms of
$$\Big(\sum_i X_i \otimes \lambda(g_i)\Big)^n = \sum_{w \in W_n^+} X^w \otimes \lambda(g)^w.$$
- Haagerup and Pisier's generalization immediately gives a lower estimates, but not upper estimates.
- Luckily there is another generalization of Haagerup's inequality due to Buchholtz in 1999 [1].

Generalizations of Haagerup's Inequality

But first some notation:

- For integer $0 \leq k \leq n$, we denote

$$\|X\|_{C^kR^{n-k}} = \| \begin{pmatrix} X_1 \\ \dots \\ X_d \end{pmatrix} \dots \begin{pmatrix} X_1 \\ \dots \\ X_d \end{pmatrix} \begin{pmatrix} X_1 & \dots & X_d \end{pmatrix} \dots \begin{pmatrix} X_1 & \dots & X_d \end{pmatrix} \|$$

  where we have k-column operators, and $n-k$ row operators.
- The multiplication of two columns, or two rows are shown by example on the next slide.

Generalizations of Haagerup's Inequality

Example: $d = 2,\ n = 3,\ k = 3$

- $$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1X_1 \\ X_2X_1 \\ X_1X_2 \\ X_2X_2 \end{pmatrix} = \begin{pmatrix} X_1X_1X_1 \\ X_2X_1X_1 \\ X_1X_2X_1 \\ X_2X_2X_1 \\ X_1X_1X_2 \\ X_2X_1X_2 \\ X_1X_2X_2 \\ X_2X_2X_2 \end{pmatrix}.$$
- The result is a column operator of all words in $d = 2$ letters of length $k = 3$, ordered according to the reversed lexicographic order on words.

Generalizations of Haagerup's Inequality

- A crucial fact for us is the submultiplicativity:

$$\|X\|_{C^k R^{n-k}} \leq \|X\|_{C^k} \, \|X\|_{R^{n-k}}.$$

- Also observe that $\|X\|_{C^n}$ and $\|X\|_{R^n}$ are column and row norms, where the tuple ranges over all reduced words of length n in generators and not there inverses.

Generalizations of Haagerup's Inequality

Theorem (Buchholtz[1]; 1999)

If $X_1, \ldots, X_d$ bounded operators on some Hilbert space $\mathcal{H}$, then for any integer $n \geq 1$

$$\| \sum_{w \in W_n^+} X^w \otimes \lambda(g)^w \| \leq (n+1) \max_{k=0}^{n} \left\{ \|X\|_{C^k R^{n-k}} \right\}$$

Remark:

1. We recover the generalized Haagerup inequality in the case $n = 1$.
2. This holds more generally for a family of "$*$-free R-Diagonal" operators in the sense of free probability. See [4] for more on this.

A Spectral Radius

Given a tuple $X_1, ..., X_d \in B(\mathcal{H})$, we can define two completely positive maps

$$\Phi_X^{\mathrm{row}}(T) := \sum_{i=1}^{d} X_i T X_i^* \quad , \text{ and } \ \Phi_X^{\mathrm{col}}(T) := \sum_{i=1}^{d} X_i^* T X_i$$

We then define

$$\rho_{\mathrm{row}}(X) := \rho(\Phi_X^{\mathrm{row}})^{1/2}, \text{ and } \ \rho_{\mathrm{col}}(X) := \rho(\Phi_X^{\mathrm{col}})^{1/2}.$$

A Spectral Radius

Both spectral radii reduce to

$$\rho_{\mathrm{row}}(X) = \lim_{n\to\infty} \| \sum_{w\in W_n^+} X^w (X^w)^* \|^{\frac{1}{2n}} = \lim_{n\to\infty} \|X\|_{R^n}^{\frac{1}{n}}$$

and

$$\rho_{\mathrm{col}}(X) = \lim_{n\to\infty} \| \sum_{w\in W_n^+} (X^w)^* X^w \|^{\frac{1}{2n}} = \lim_{n\to\infty} \|X\|_{C^n}^{\frac{1}{n}}$$

since the norms are realized on the identity.

Outer Spectral Radius: Matrix Tuples

For general $X_1, ..., X_d \in B(\mathcal{H})$ equality of $\rho_{\mathrm{row}}(X)$ and $\rho_{\mathrm{col}}(X)$ fails, but for $\dim(H) < \infty$ we have equality.

Theorem (Pascoe [8]; 2021)

Let $X_1, ..., X_d \in M_\ell(\mathbb{C})$, *then*

$$\lim_{n\to\infty} \|X\|_{C^n}^{\frac{1}{n}} = \lim_{n\to\infty} \|X\|_{R^n}^{\frac{1}{n}} = \rho\Big(\sum_i \overline{X_i} \otimes X_i\Big)^{\frac{1}{2}}$$

- where $\rho\left(\sum_i \overline{X_i} \otimes X_i\right)^{\frac{1}{2}}$ is called the **outer spectral radius** of the tuple $(X_1, ..., X_d)$.
- The outer spectral radius in the case $\dim(\mathcal{H}) = \infty$ have been studied independently by other authors in [2], [9], and [10].

Main Result

Below the spectral radius is taken in the minimal tensor product $B(\mathcal{H}) \otimes C^*_\lambda(\mathbb{F}_d)$.

Theorem (Jury-JvR; 2025)

Let $X_1, \dots, X_d$ *be bounded operators on some Hilbert space* $\mathcal{H}$ *. Then*

$$\rho\Big(\sum_i X_i \otimes \lambda(g_i) \Big) = \max\Big\{ \rho_{col}(X), \ \rho_{row}(X) \Big\}$$

Corollary

Let $X_1, \dots, X_d \in M_\ell(\mathbb{C})$. *Then*

$$\rho\Big(\sum_i X_i \otimes \lambda(g_i) \Big) = \rho\Big(\sum_i \overline{X_i} \otimes X_i \Big)^{\frac{1}{2}}$$

Proof Sketch

Proof (matrix case):
Recall Gelfand's spectral radius formula

$$\rho\Big(\sum_i X_i \otimes \lambda(g_i)\Big) = \lim_{n\to\infty} \| \sum_{w\in W_n^+} X^w \otimes \lambda(g)^w \|^{1/n}.$$

The **lower** bound follows immediately from Haagerup-Pisier's [6] lower bound. For the **upper** bound set $\rho\Big(\sum_i \overline{X_i} \otimes X_i\Big) = 1$, and let $\epsilon > 0$. It suffices to show

$$\rho\Big(\sum_i X_i \otimes \lambda(g_i)\Big) \leq 1 + \epsilon.$$

Proof Sketch

Since both $\|X\|_{C^n}^{1/n}, \ \|X\|_{R^n}^{1/n} \to 1$ choose $N > 0$ such that

$$\sup_{n \geq N} \left\{ \ \|X\|_{C^n}^{1/n}, \ \|X\|_{R^n}^{1/n} \ \right\} \leq 1 + \epsilon$$

And choose $M \geq 1 + \epsilon$ so that

$$\max_{1 \leq n \leq N-1} \left\{ \ \|X\|_{C^n}, \ \|X\|_{R^n} \ \right\} \leq M.$$

Proof Sketch

Construct two sequence $k(n)$ and $j(n)$ by choosing $0 \leq k(n) \leq n$ such that

$$\|X\|_{C^{k(n)}R^{n-k(n)}} = \max_{0\leq k\leq n}\left\{ \|X\|_{C^kR^{n-k}} \right\}.$$

Let $j(n) = \min\{k(n),\ n-k(n)\}$. Now either $j(n)$ is **bounded** or **unbounded** as $n \to \infty$, and we can split into two cases.

Proof Sketch

Case 1: Suppose $\sup_n j(n) \leq K < +\infty$.

For $n \geq N + K$ we claim

$$\|X\|_{C^{k(n)}R^{n-k(n)}} \leq M(1+\epsilon)^n.$$

Assuming the claim, we can apply Buchholtz's result, and take n^{th} roots to obtain

$$\| \sum_{w \in W_n^+} X^w \otimes \lambda(g)^w \|^{\frac{1}{n}} \leq ((n+1)M)^{\frac{1}{n}} (1+\epsilon).$$

The result for case 1 follows by taking $n \to \infty$.

Proof Sketch

Case 1: Firstly, if $j(n) = k(n)$, then we have $n - k(n) \geq N$, and

$$\|X\|_{C^{k(n)}} \|X\|_{R^{n-k(n)}} \leq M(1+\epsilon)^{n-k(n)} \leq M(1+\epsilon)^n.$$

Otherwise, if $j(n) = n - k(n)$, then $k(n) \geq N$ and we have

$$\|X\|_{C^{k(n)}} \|X\|_{R^{n-k(n)}} \leq (1+\epsilon)^{k(n)} M \leq (1+\epsilon)^n M$$

as required to complete the claim and case 1.

Proof Sketch

Case 2: Suppose $\sup_n j(n) = +\infty$.

Choose a subsequence $j(n_p) \to \infty$. Then both n_p and $n_p - k(n_p) \geq N$ for sufficiently large p. Hence

$$\|X\|_{C^{k(n_p)} R^{n_p - k(n_p)}} \leq (1+\epsilon)^{k(n_p)} (1+\epsilon)^{n_p - k(n_p)} = (1+\epsilon)^{n_p}.$$

And complete case 2 similarly to case 1. □

Other Spectral Radii

The spectral radius of linear pencils can also be computed for other families of $*$-free elements. In particular, replacing the upper bound in [1] with de la Salle's version of the Haagerup inequality [4] shows that free Haar unitaries may be replaced by "free circular" elements.

Theorem (Jury-JvR; 2025)

Let $X_1, ..., X_d$ *be bounded operators on some Hilbert space* $\mathcal{H}$*, and* $c_1, \cdots, c_g$ $*$*-free circular elements. Then*

$$\rho\Big(\sum_i X_i \otimes c_i \Big) = \max\Big\{ \rho_{col}(X), \ \rho_{row}(X) \Big\}$$

Thank You!

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