The Spectral Radius of Free Haar Unitary Pencils

Vikus J. v. Rensburg
Joint work with Mike Jury

University of Florida Analysis and Probability Seminar
September, 2025

Outline

Introduction and Motivation

Introduction

- Fix an integer $d \geq 1$.
- Let $\mathbb{F}_d$ denote the **free group** on d generators $\{g_1, g_2, \ldots, g_d\}$.
- Denote by

$$\lambda : \mathbb{F}_d \to \mathcal{U}(\ell^2(\mathbb{F}_d))$$

  the **left regular representation** of the free group.
- The **reduced C^*-algebra**

$$C^*_\lambda(\mathbb{F}_d)$$

  is the C^*-algebra generated by $\lambda(\mathbb{F}_d)$.

Introduction

- The image of the generators under λ is called **the free Haar unitary** generators, and denoted $u_i := \lambda(g_i)$.
- Fix $X_1, \ldots, X_d \in M_\ell(\mathbb{C})$. By a **linear pencil** in free Haar unitaries we mean

$$X_1 \otimes u_1 + X_2 \otimes u_2 + \ldots + X_d \otimes u_d.$$

- For example if $X_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, then

$$X_1 \otimes u_1 + X_2 \otimes u_2 = \begin{bmatrix} u_1 & u_1 + u_2 \\ u_2 & 2u_2 \end{bmatrix}$$

Preliminaries

Notation:

- Given a reduced word $w = g_{i_1} \dots g_{i_k}$ (not inverses) we denote
$$X^w = X_{i_1} \dots X_{i_k}$$
and
$$\lambda(g)^w = \lambda(g_{i_1}) \dots \lambda(g_{i_k}).$$
- For example if $d = 3$, and $w = g_2 g_1 g_3 g_3$, then
$$X^w = X_2 X_1 X_3 X_3.$$
- We denote the **length** of the reduce word by $|w|$.

Motivation

- A central question in Random Matrix Theory is to study the eigenvalues of a random matrix.
- In an ongoing project by Mike Jury and George Roman, they are interested in the eigenvalues of random unitary pencils

$$X_1 \otimes U_1^{(n)} + \cdots + X_d \otimes U_d^{(n)}$$

as $n \to \infty$.
- Here, $U_i^{(n)}$ are independent, $n \times n$, Haar-distributed, unitary random matrices, and $X_i \in M_\ell(\mathbb{C})$ are fixed coefficients.

Motivation

- There is a sense in which the deterministic operator

$$X_1 \otimes u_1 + X_2 \otimes u_2 + \cdots + X_d \otimes u_d$$

is the limiting object of the random unitary pencil.

- This notion is made precise by the work of Voiculescu in 1991 [9], and later developed by Collins and Male [3] in 2014.
- Our approach to understanding the eigenvalues of the large random matrices is to study the spectrum of this deterministic operator.

Main Question

Can we find a formula for the spectral radius

$$\rho(\sum_i X_i \otimes u_i)$$

only in terms of the coefficient matrices $X_1, ..., X_d$?

Haagerup's Inequality

Classical Haagerup's Inequality

- In 1979, Haagerup [4] proved

$$\| \sum_{g \in W_n} \alpha_g \lambda(g) \| \leq (n+1)(\sum_g |\alpha_g|^2)^{\frac{1}{2}}$$

for any finitely supported $(\alpha_g) : \mathbb{F}_g \to \mathbb{C}$, where $W_n \subset \mathbb{F}_g$ of all reduced words of length n.

Generalized Haagerup's Inequality

- In 1993, Haagerup and Pisier [5] replaced $\mathbb{C}$ with an arbitrary C^*-algebra $\mathcal{A}$.
- They showed

$$\| \sum_g \alpha_g \otimes \lambda(g) \| \geq \max\{\| \sum_g \alpha_g^* \alpha_g \|^{\frac{1}{2}}, \| \sum_g \alpha_g \alpha_g^* \|^{\frac{1}{2}}\}$$

for any finitely supported $(\alpha_g) : \mathbb{F}_d \to \mathcal{A}$.

- Also if (α_g) is supported on $\{g_1, .., g_d, g_1^{-1}, .., g_d^{-1}\}$, then

$$\| \sum_g \alpha_g \otimes \lambda(g) \| \leq 2 \max\{\| \sum_g \alpha_g^* \alpha_g \|^{\frac{1}{2}}, \| \sum_g \alpha_g \alpha_g^* \|^{\frac{1}{2}}\}$$

Generalizations of Haagerup's Inequality

- By Gelfand's Formula we need to estimate n^{th} roots of norms of
$$\Big(\sum_i X_i \otimes \lambda(g_i)\Big)^n = \sum_{|w|=n} X^w \otimes \lambda(g)^w.$$
- Haagerup and Pisier's generalization immediately gives a lower estimates, but not upper estimates.
- Luckily there is another generalization of Haagerup's inequality due to Buchholtz in 1999 [1].

Generalizations of Haagerup's Inequality

Notation:

- For integer $0 \leq k \leq n$, we denote

$$\|X\|_{C^k R^{n-k}} = \| \begin{pmatrix} X_1 \\ \dots \\ X_d \end{pmatrix} \dots \begin{pmatrix} X_1 \\ \dots \\ X_d \end{pmatrix} \begin{pmatrix} X_1 & \dots & X_d \end{pmatrix} \dots \begin{pmatrix} X_1 & \dots & X_d \end{pmatrix} \|$$

  where we have k-column operators, and $n-k$ row operators.
- The multiplication of two columns, or two rows are shown by example on the next slide.

Generalizations of Haagerup's Inequality

Example: $d = 2$, $n = 3$, $k = 3$

- $$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1X_1 \\ X_1X_2 \\ X_2X_1 \\ X_2X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1X_1X_1 \\ X_1X_1X_2 \\ X_1X_2X_1 \\ X_1X_2X_2 \\ X_2X_1X_1 \\ X_2X_1X_2 \\ X_2X_2X_1 \\ X_2X_2X_2 \end{pmatrix}.$$
- The result is a column operator of all words in $d = 2$ letters of length $k = 3$, ordered according to the lexicographic order on words.

Generalizations of Haagerup's Inequality

- A crucial fact for us is the "submultiplicativity":

$$\|X\|_{C^k R^{n-k}} \leq \|X\|_{C^k} \, \|X\|_{R^{n-k}}.$$

- Also observe that $\|X\|_{C^n}$ and $\|X\|_{R^n}$ are column and row norms, where the tuple ranges over all reduced words of length n.

Generalizations of Haagerup's Inequality

Theorem (Buchholtz[1]; 1999)

If $X_1, \ldots, X_d \in M_\ell(\mathbb{C})$, *then*

$$\|\sum_{|w|=n} X^w \otimes \lambda(g)^w\| \leq (n+1)\max_{k=0}^{n}\left\{\|X\|_{C^k R^{n-k}}\right\}$$

Remark:

1. We recover the generalized Haagerup inequality in the case $n = 1$.
2. This holds for more generally for $X_i \in \mathcal{A}$, which we come back to later.

Spectral Radius

Joint Spectral Radius

Given a tuple $X_1, ..., X_d \in \mathcal{A}$, we can define two completely positive maps

$$\Phi_X^{\mathrm{row}}(T) := \sum_{i=1}^{d} X_i T X_i^* \quad , \text{ and } \quad \Phi_X^{\mathrm{col}}(T) := \sum_{i=1}^{d} X_i^* T X_i$$

We then define

$$\rho_{\mathrm{row}}(X) := \rho(\Phi_X^{\mathrm{row}})^{1/2}, \text{ and } \quad \rho_{\mathrm{col}}(X) := \rho(\Phi_X^{\mathrm{col}})^{1/2}.$$

Joint Spectral Radius

Both spectral radii reduce to

$$\rho_{\mathrm{row}}(X) = \lim_{n\to\infty} \| \sum_{|w|=n} X^w (X^w)^* \|^{\frac{1}{2n}}$$

and

$$\rho_{\mathrm{col}}(X) = \lim_{n\to\infty} \| \sum_{|w|=n} (X^w)^* X^w \|^{\frac{1}{2n}}$$

since the norms are realized on the identity.

Joint Spectral Radius

The following authors have independently established results characterizing similarity to a strict row contraction via the joint spectral radius:

- Bunce (1984) [2]
- Popescu (2014) [7]
- Pascoe (2021) [6]
- Shalit and Shamovich (2025) [8]

Outer Spectral Radius: Matrix Tuples

For general $X_1, \ldots, X_d \in \mathcal{A}$ equality of $\rho_{\mathrm{row}}(X)$ and $\rho_{\mathrm{col}}(X)$ fails, but for $\dim(H) < \infty$ we have equality.

Theorem (Pascoe[6]; 2021)

Let $X_1, \ldots, X_d \in M_\ell(\mathbb{C})$, then

$$\lim_{n\to\infty} \|X\|_{C^n}^{\frac{1}{n}} = \lim_{n\to\infty} \|X\|_{R^n}^{\frac{1}{n}} = \rho\Big(\sum_i \overline{X_i} \otimes X_i\Big)^{\frac{1}{2}}$$

where $\rho\left(\sum_i \overline{X_i} \otimes X_i\right)^{\frac{1}{2}}$ is called the **outer spectral radius** of the matrix tuple $(X_1, \ldots, X_d)$.

Main Result

Main Result

Theorem (Jury-JvR; 2025)

Let $X_1, \ldots, X_d \in M_\ell(\mathbb{C})$. *Then*

$$\rho(\sum_i X_i \otimes \lambda(g_i)) = \rho(\sum_i \overline{X_i} \otimes X_i)^{\frac{1}{2}}$$

Proof Sketch

The lower bound follows from Haagerup-Pisier generalization of the inequality, and the fact that both column norm $\|X\|_{C^n}^{\frac{1}{n}}$ and row norm $\|X\|_{R^n}^{\frac{1}{n}}$ converge to the outer spectral radius of $(X_1, .., X_d)$.

For the upper bound, it suffices to prove if

$$\rho\Big(\sum_i \overline{X_i} \otimes X_i\Big)^{1/2} < 1,$$

then

$$\rho\Big(\sum_i X_i \otimes \lambda(g_i)\Big) < 1.$$

Proof Sketch

Denote

$$r = \rho\Big(\sum_i \overline{X_i} \otimes X_i\Big)^{1/2},$$

and choose t such that $r < t < 1$.
There exists $N > 0$ such that both

$$\|X\|_{C^n}^{\frac{1}{n}}, \ \|X\|_{R^n}^{\frac{1}{n}} \leq t, \quad n \geq N.$$

Choose $M \geq 1$ so that

$$\max_{1 \leq n \leq N-1} \{\|X\|_{C^n}, \|X\|_{R^n}\} \leq M.$$

Proof Sketch

For each n, choose $k(n) \in \{0, \ldots, n\}$ such that

$$\|X\|_{C^{k(n)}R^{n-k(n)}} = \max_{0 \leq k \leq n} \left\{\|X\|_{C^k R^{n-k}}\right\}.$$

Let $j(n) = \min\{k(n),\ n - k(n)\}$. Now either $j(n)$ is bounded or not bounded as $n \to \infty$, and we can split into two cases.

Proof Sketch

Case 1: $j(n)$ bounded.

Suppose $\sup_n j(n) \leq K < +\infty$. For fixed $n \geq N + K$ we claim

$$\|X\|_{C^{k(n)}R^{n-k(n)}} \leq Mt^{n-K}.$$

Assuming the claim, we can apply Buchholtz's result, and take n^{th} roots to obtain

$$\|\sum_{|w|=n} X^w \otimes \lambda(g)^w\|^{\frac{1}{n}} \leq t\Big((n+1)Mt^{-K}\Big)^{\frac{1}{n}}.$$

The result for case 1 follows by taking $n \to \infty$.

Proof Sketch

Firstly, if $j(n) = k(n)$, then we have $t^{n-k(n)} \leq t^{n-K}$, and

$$\|X\|_{C^{k(n)}}\|X\|_{R^{n-k(n)}} \leq Mt^{n-k(n)} \leq Mt^{n-K}.$$

Secondly, if $j(n) = n - k(n)$, then $n - k(n) \leq K$ implies $t^{k(n)} \leq t^{n-K}$, and we have

$$\|X\|_{C^{k(n)}}\|X\|_{R^{n-k(n)}} \leq t^{k(n)}M \leq Mt^{n-K}.$$

Proof Sketch

Case 2: $j(n)$ unbounded.

Suppose $\sup_n j(n) = +\infty$. Choose a subsequence $j(n_p) \to \infty$. Then

$$\|X\|_{C^{k(n_p)}R^{n_p-k(n_p)}} \leq t^{k(n_p)} t^{n_p-k(n_p)} = t^{n_p}.$$

By Buchholtz,

$$\| \sum_{|w|=n_p} X^w \otimes \lambda(g)^w \|^{\frac{1}{n_p}} \leq \left((n_p+1)\|X\|_{C^{k(n_p)}R^{n_p-k(n_p)}}\right)^{\frac{1}{n_p}} \leq t\left(n_p+1\right)^{\frac{1}{n_p}}.$$

And the result follows since every subsequence of a convergent sequence has the same limit.

Proof Sketch

Lastly, the upper bound follows from applying the above claim to the tuple

$$Y_i := \frac{1}{\alpha + \epsilon} X_i,$$

where

$$\alpha := \rho(\sum_i \overline{X_i} \otimes X_i)^{\frac{1}{2}},$$

and homogeneity of the spectral radius. □

Operator Coefficients

For a general C^*-algebra $\mathcal{A}$, we can use similar techniques to show:

Theorem (Jury-JvR; 2025)

Let $X_1, \ldots, X_d \in \mathcal{A}$. Then

$$\rho(\sum_i X_i \otimes \lambda(g_i)) = \max\left\{\rho_{col}(X),\ \rho_{row}(X)\right\}$$

Remark: We lose equality of $\rho_{col}(X)$, and $\rho_{row}(X)$ in the general case.

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