Uniform Bounds for Expected Determinants

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Haar Measure

- We denote the group of $d \times d$ unitary matrices by $\mathcal{U}(d)$, and the product space $\mathcal{U}(d)^g$.
- Recall that $\mathcal{U}(d)$ is equipped with a unique, translation invariant probability measure namely **normalized Haar measure**.
- This is an analog of surface area measure on the sphere.
- Another way to interpret Haar measure is to consider a random unitary conjugation which is a random change of basis, where the randomness is distributed uniformly over $\mathcal{U}(d)$.

Random Matrices

1. By an $n \times n$ **random matrix** X we mean a matrix-valued random variable $X : \Omega \to M_n(\mathbb{C})$.
2. By a $d \times d$ **Haar distributed random unitary** we mean a unitary valued random matrix with distribution given by normalized Haar measure on the unitary group.
3. **Notation:** When we want to emphasize the size of the Haar unitaries we write $U_i^{(d)}$ or $U^{(d)} = (U_1^{(d)}, \cdots, U_g^{(d)})$. The latter when we consider a g-tuple of $d \times d$ classically independent Haar unitary matrices.

Large Dimensional Limit

Theorem (Jury-Roman; 2024)

Let $r < 1$, and $k \in \mathbb{N}$. Suppose there exists $C(k, r, g) > 0$ s.t.

$$\sup_{d \in \mathbb{N}} \mathbb{E}\Big[\, |\det L_X(U^{(d)})|^2 \, \Big] \leq C(k, r, g) \tag{1}$$

for all $X \in M_k(\mathbb{C})^g$ with $\|X\|_{row} \leq r$.

Then for any $Y \in M_\ell(\mathbb{C})^g$, and $Z \in M_{\ell'}(\mathbb{C})^g$ with $\rho_{out}(Y) < 1$ and $\rho_{out}(Z) < 1$

$$\lim_{d \in \mathbb{N}} \mathbb{E}\Big[\det L_Y(U^{(d)}) \, \det L_Z(U^{(d)})^* \Big] = \det\Big(I_\ell \otimes I_{\ell'} - \sum_i Y_i \otimes \overline{Z}_i \Big)^{-1} \tag{2}$$

Today's Goal

The hypotheses in (1) were previously shown to hold under the additional assumption that the coefficient tuples X are upper triangular. We now remove this assumption as a consequence of:

Theorem (Jury-Roman-v.R.; 2025)

Let $r < 1$, $k \in \mathbb{N}$, and $U_1^{(d)}, \ldots, U_g^{(d)}$ denote i.i.d. Haar distributed random unitaries. Then there exists a constant $C(k,r,g)$ such that

$$\sup_{d \in \mathbb{N}} \mathbb{E}\Big[| \det(I_k \otimes I_d + \sum_{i=1}^{g} X_i \otimes U_i^{(d)}) |^2 \Big] \leq C(k,r,g)$$

for all $X = (X_1, \ldots, X_g) \in M_k(\mathbb{C})^g$ such that $\|X\|_{row} \leq r$.

The Toolbox

Concentration of Measure

Equip $\mathcal{U}(d)^g$ with the ℓ^2 HS-metric $\mathrm{d}(U,V) := \left(\sum_{i=1}^{g} \|U_i - V_i\|_2^2\right)^{\frac{1}{2}}$.

Theorem (Meckes–Meckes; 2013 [6])

Let $F : \mathcal{U}(d)^g \to \mathbb{R}$ be a L-Lipschitz function denote $U_1, \ldots, U_g$ independent Haar-distributed $d \times d$ unitary random matrices. Then

$$\mathbb{P}\left(|F(U_1, \ldots, U_g)| \geq t \right) \leq 2\exp\left(-\frac{d\left(t - |\mathbb{E}F(U_1, \ldots, U_g)|\right)^2}{24L^2} \right).$$

for every $t > |\mathbb{E}F(U_1, \ldots, U_g)|$.

Random Matrices and Deterministic Operators

- Haar-distributed unitary matrices appear in many branches of mathematics, ranging from number theory (the Riemann zeta function [5]) and combinatorics to complex analysis (the strong Szegő theorem [1]) and operator algebras (asymptotic freeness [10], [9]).
- In this work, we are interested in the latter connection, established by Voiculescu in 1991 ([10]).
- Voiculescu showed that his theory of free probability provides a natural limiting candidate for g classically independent Haar random unitaries in the large-dimension limit.
- This result has been refined and extended over the past thirty years by Collins–Male in 2014 ([2]) and Parraud in 2022 ([8]).

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Free Haar unitary operators

- Let $\mathbb{F}_g$ denote the **free group** on g generators.
- Denote by
$$\lambda : \mathbb{F}_g \to \mathcal{U}(\ell^2(\mathbb{F}_g))$$
the **left regular representation** of the free group.
- We use τ to denote the unique tracial state on the free group factor $L(\mathbb{F}_g)$.
- We write $u_1, \ldots, u_g$ for the images of the generators under the left-regular representation, yielding a concrete model of Voiculescu's free Haar unitary family.

Refined Strong Convergence I

A consequence of Voiculescu's initial work in [10] is that for any nc-polynomial $p \in M_k(\mathbb{C})\langle x, x^* \rangle$ we have

$$\frac{1}{kd}\mathbb{E}\left[\mathrm{Tr}_{kd}(p(U^{(d)}, U^{(d)*})\right] \to (\frac{1}{k}\mathrm{Tr}_k \otimes \tau)(\ p(u, u^*)\ ) \qquad (3)$$

as $d \to \infty$.

Some follow up questions:

1. What about more general nc-functions?
2. What is the rate of convergence?
3. Higher order asymptotics?

Refined Strong Convergence II

Theorem (Parraud; 2022 [8])

Let $U_1^{(d)}, \cdots, U_g^{(d)}$ be independent, $d \times d$, Haar distributed, random unitary matrices, and $u_1, \cdots, u_g$ free Haar unitary operators. Suppose $p \in M_k(\mathbb{C})\langle x, x^*\rangle$ is a formally self-adjoint nc-polynomial, and $f : \mathbb{R} \to \mathbb{R}$ at least 6 times differentiable. Then there exists a constant $C(p) > 0$ (only depending on p) such that

$$\left|\frac{1}{kd}\mathbb{E}\left[Tr_{kd}\, f(p(U^{(d)}, U^{(d)*})) - (\frac{1}{k}Tr_k \otimes \tau)f(p(u, u^*))\right]\right| \leq \frac{\log(d)^2 k^2}{d^2} C(p)\|f\|_{C^6} \tag{4}$$

for all $d \in \mathbb{N}$.

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for all $d \in \mathbb{N}$.

Refined Strong Convergence III

Our Setting:

We will have $(\frac{1}{k}\mathrm{Tr}_k \otimes \tau) f(p(u, u^*)) = 0$, so that Parraud's estimates give

$$\begin{aligned} \left|\mathbb{E}\left[\mathrm{Tr}_{kd}\, f(p(U^{(d)}, U^{(d)*}))\right]\right| \\ \leq \frac{\log(d)^2 k^3}{d} C(p) \|f\|_{C^6} \qquad (5) \\ \to 0 \end{aligned}$$

as $d \to \infty$.

Spectral Radius Formula

Theorem (Jury-v.R.; 2025)

Let $X_1, \ldots, X_g \in M_k(\mathbb{C})$, and let $u_1, u_2, \ldots, u_g$ be free Haar unitaries. Then

$$\rho\left(\sum_{i=1}^{g} X_i \otimes u_i\right) = \rho\left(\sum_{i=1}^{g} X_i \otimes \overline{X_i}\right)^{1/2}.$$

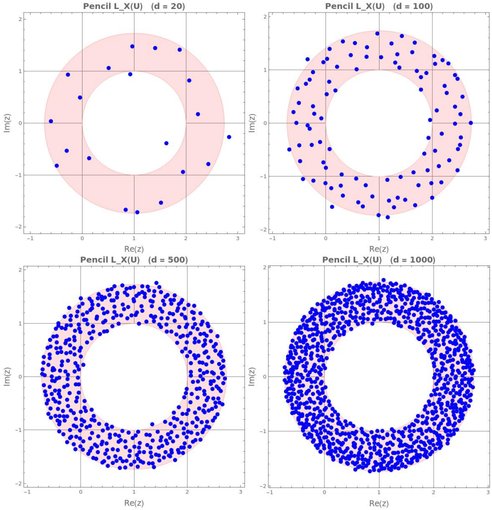
Remark:

The quantity $\rho\left(\sum_{i=1}^{g} X_i \otimes \overline{X_i}\right)^{1/2}$ is called the **outer spectral radius** of the tuple X, and denoted $\rho_{\text{out}}(X)$.

"Leaky" Spectrum

Example:

- $L_X(u) = 1 + \sqrt{2}u_1 + u_2$
- $\sigma(L_X(u)) =$ $\mathrm{An}(1, 1, \sqrt{3})$ by [3].
- Eigen values of $L_X(U^{(d)})$ in blue.



Lipschitz Constants

Proposition

Suppose $p = \sum_{w \in \mathbb{F}_g} c_w x^w$ *with coefficients* $c_w \in M_k(\mathbb{C})$*, and at most finitely many non-zero. Then evaluation of* p *induces a Lipschitz function on* $\mathcal{U}(d)^g$ *with constant*

$$L_p := \sum_{w \in \mathbb{F}_g} \|c_w\|_2 |w|.$$

Proof Sketch

Main Idea:
By using a telescopic sum, and the fact that the HS-norm is invariant under unitary multiplication one can show that

$$\|U^w - V^w\|_2 \leq |w| \mathrm{d}(U,V)$$

where $\mathrm{d}(U,V) := (\sum_{i=1}^{g} \|U_i - V_i\|_2)^{\frac{1}{2}}$. (Do quick example on the board.)

Main Results

Uniform Tail Bounds

Theorem (Jury-Roman-v.R.; 2025)

Given a formally self-adjoint, nc-polynomial $p \in M_k(\mathbb{C})\langle x, x^* \rangle$.

Suppose

1. $p(u, u^*)$ *is positive and invertable in* $M_k(\mathbb{C}) \otimes L(\mathbb{F}_g)$ *and*
2. $(\frac{1}{k} Tr_k \otimes \tau)(\log\ p(u, u^*)) = 0$.

Let $[a, b] \subset (0, \infty)$ *denote an interval containing the spectrum of* $p(u, u^*)$, *and* $f : \mathbb{R} \to \mathbb{R}$ *a* C^∞ *function that agrees with* $\log(x)$ *on* $[a, b]$.

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Uniform Tail Bounds

Theorem (continued)

Consider the random variable

$$F_d := Tr_{kd} f(\ p(U^{(d)}, U^{(d)*})\ ).$$

Then there exists constants $c, C > 0$ *(only dependent on* p *and* f*) such that for every* $t > 0$, $d \in \mathbb{N}$,

$$\mathbb{P}\Big(|F_d| > t\Big) \leq C \exp(-ct^2) \qquad (*)$$

Proof Sketch: Concentration Step I

The map

$$U \longmapsto \mathrm{Tr}_{kd}(f(p(U, U^*))), \quad U \in \mathcal{U}(d)^g$$

is Lipschitz with constant $L\sqrt{d}$ (only $\sqrt{d}$!), where $L > 0$ depends only on f and p .

1. p is a polynomial, hence Lipzschitz.
2. There exists a $K > 0$ such that $\sup_d \|p(U^{(d)}, U^{(d)})\| \leq K$. Since f is smooth, we know f is Lipschitz on $[-K, K]$.
3. The Tr_{kd} is Lipzschitz with constant $\sqrt{kd}$.

Proof Sketch: Concentration Step II

It follows from concentration of measure for Haar unitary matrices [7] that for $t \geq |\mathbb{E}F_d|$

$$\mathbb{P}\Big(|F_d| > t\Big) \leq 2\exp\Big(\frac{-d\,(t - |\mathbb{E}F_d|)^2}{24(L\sqrt{d})^2}\Big) \leq 2\exp(-C_0(t - |\mathbb{E}F_d|)^2) \tag{I}$$

for constant $C_0 = \frac{1}{24L^2} > 0$.

Proof Sketch: Strong Convergence Step

Since f agrees with $\log(x)$ on the spectrum of $p(u, u^*)$, and $\log\ p(u, u^*)$ is centered, it follows from Parraud's estimates [8]

$$|\mathbb{E} F_d| \leq \frac{\log(d)^2 k^3}{d} \cdot C(p) \cdot \|f\|_{C^6}$$

Hence

$$M := \sup_d |\mathbb{E} F_d| < +\infty.$$

Proof Sketch

As a result we have

$$\begin{aligned}
&\mathbb{P}(|F_d| > t) \\
&\leq 2\exp\left(-C_0(t - |\mathbb{E}F_d|)^2\right) \quad \text{, by concentaration} \\
&\leq 2\exp\left(-C_0(t - M)^2\right) \quad \text{, by strong convergence.}
\end{aligned}$$

for all d, and $t > M$. $\square$

Consequence of Uniform Tail Bounds

We restate our main goal for today.

Corollary

Let $r < 1$, $k \in \mathbb{N}$, and $U_1^{(d)}, \dots, U_g^{(d)}$ denote i.i.d. Haar distributed random unitaries,. Then there exists a constant $C(k, r, g)$ such that

$$\sup_{d \in \mathbb{N}} \mathbb{E}\left[\left| \det(I_k \otimes I_d + \sum_{i=1}^{g} X_i \otimes U_i^{(d)}) \right|^2 \right] \leq C(k, r, g)$$

for all $X = (X_1, \dots, X_g) \in M_k(\mathbb{C})^g$ such that $\|X\|_{row} \leq r$.

Proof: Spectral Radius Step I

Since $\rho_{\text{out}}(X) < 1$, it follows from the spectral radius formula that

$$\rho\left(\sum_i X_i \otimes u_i\right) = \rho_{\text{out}}(X) < 1.$$

Thus

$$\sum_i X_i \otimes u_i$$

is invertible, and hence

$$\sigma(L_X(u)L_X(u)^*) \subset [a, b] \quad \text{for some } b > a > 0.$$

Proof: Spectral Radius Step II

Also since

$$\rho\left(\sum_i X_i \otimes u_i\right) < 1,$$

it follows from expanding $\log(L_X(u))$ as a power series, and the fact that

$$\left(\frac{1}{k}\mathrm{Tr} \otimes \tau\right)\left(\left(\sum_i X_i \otimes u_i\right)^n\right) = 0 \quad \text{for } n \geq 1,$$

that

$$\left(\frac{1}{k}\mathrm{Tr} \otimes \tau\right)(\log(L_X(u)L_X(u)^*)) = 0.$$

Proof: Uniform Tail Bound Application I

Now let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

1. $f(x) = \log(x) \quad$ for $x \in [a, b]$,
2. $f(x) \geq \log(x) \quad$ for $x > 0$.

Thus,

$$\left(\frac{1}{k}\mathrm{Tr} \otimes \tau\right)(f(L_X(u)L_X(u)^*)) = 0.$$

And

$$\mathrm{Tr}_{kd}\left[f\left(L_X(U^{(d)})L_X(U^{(d)})^*\right)\right]$$

satisfies all assumptions of the uniform tail bound theorem.

Proof: Uniform Tail Bound Application II

- With the convention $\log(0) := -\infty$, and $\exp(-\infty) := 0$ it follows that

$$\det(A) = \exp(\operatorname{Tr}\log(A))$$

for any positive semi-definite matrix A with 0 possibly an eigen value.

- Since $\log \leq f$

$$\det(A) = \exp(\operatorname{Tr}\log(A)) \leq \exp(\operatorname{Tr} f(A))$$

Proof: Uniform Tail Bound Application III

- By an application of the uniform tail bounds there exists constants $c, C > 0$ (dependent on k, r, g and f) such that

$$\mathbb{P}\left(\left| \mathrm{Tr}_{kd}\, f(L_X(U^{(d)})L_X(U^{(d)})^*) \right| > t \right) \leq c\exp(-Ct^2)$$

for any $t > 0$, $d \in \mathbb{N}$, $\|X\|_{row} \leq r$, and $X \in M_k(\mathbb{C})^g$.

- Since the set of $X \in M_k(\mathbb{C})^g$ such that $\|X\|_{row} \leq r$ is compact, we can choose a single pair constants c and C.

Proof: Uniform Tail Bound Application IV

- By our sub-Gaussian bound on F_d we have

$$\mathbb{E}\Big[\det(L_X(U^{(d)})L_X(U^{(d)})^*)\Big]$$

$$\leq \int_0^\infty e^t\,\mathbb{P}\Big(\,\Big|\mathrm{Tr}_{kd}\,f(L_X(U^{(d)})L_X(U^{(d)})^*)\Big| > t\,\Big)\,dt \qquad (6)$$

$$\leq c\int_0^\infty e^t e^{-Ct^2}dt < \infty.$$

where $c, C > 0$ independent of d, and only depends on k, r and g.
□

Consequence of Uniform Tail Bounds

Corollary

If X, Y are g-tuples of square matrices of size $k \times k, \ell \times \ell$ with outer spectral radii $\rho_{out}(X), \rho_{out}(Y) < 1$, then

$$\lim_{d\to\infty} \mathbb{E}\Big[\det L_X(U^{(d)}) \det L_Y(U^{(d)})^*\Big]$$

$$= \det\left(I_k \otimes I_\ell - \sum_{i=1}^{g} X_i \otimes \overline{Y_i}\right)^{-1}.$$

Proof: Using the above corollary, this is an immediate consequence of work done by Jury-Roman in [4, Proposition 3.5]. □

Thank You!

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