CRASH COURSE CW-COMPLEXES

Abstract. The goal of these notes is to learn the basics of CW-complexes needed to understand the Euler characteristic of the surfaces constructed in [MP19]. In algebraic topology, both simplicial and CW complexes are used to construct complex topological spaces from simpler building blocks. The fundamental difference lies in their rigidity versus flexibility. The summary below is taken from here.

1. Introduction

In algebraic topology, both simplicial and CW complexes are used to construct complex topological by gluing together simplices. The fundamental difference lies in their rigidity versus flexibility. The summary below is taken from here.

Simplicial Complex.

- **Building Blocks:** Simplices (0-simplices as points, 1-simplices as lines, 2-simplices as triangles, etc.).
- **Gluing Rules:** Highly rigid. Simplices must intersect perfectly along their shared sub-faces (e.g., exactly edge-to-edge or vertex-to-vertex). Intuitively, this means that your complex is a union of simplices which are glued together by just gluing their faces together in the "obvious" linear way (for instance, as one encounters in a triangulation of a surface), rather than by arbitrary complicated continuous maps.
- **Efficiency:** Low. Requires many blocks to approximate curved spaces.
- **Best For:** Combinatorial topology, computational geometry, and algorithmic implementation.

CW Complex.

- **Building Blocks:** n-cells (spaces homeomorphic to closed n-dimensional balls or disks).
- **Gluing Rules:** Highly flexible. The boundary of an n-cell can be attached to the existing lower-dimensional skeleton using any continuous map. Must be constructed inductively. That is at each step, you add a new simplex by gluing its boundary to the complex you have already. You're not allowed to glue in a new 1 -simplex once you've already added a 2 -simplex.
- **Efficiency:** High. Highly optimized for creating spaces with minimal building blocks.
- **Best For:** Homotopy theory and efficient algebraic calculations (like cellular homology).

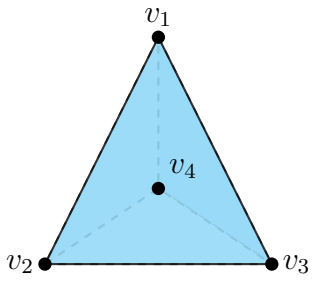
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Example 1.1 (Example: Constructing a 2-Sphere (S^2))**.**

- **Simplicial approach:** Requires a minimum of 4 vertices, 6 edges, and 4 triangular faces (structurally a hollow tetrahedron).
- **CW approach:** Requires exactly 1 vertex (0-cell) and 1 topological disk (2-cell). The entire boundary of the disk is collapsed and glued into the single vertex.

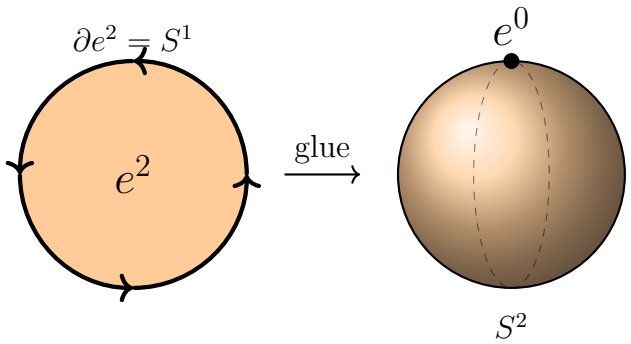
Simplicial Complex

4 Vertices, 6 Edges, 4 Faces
Strict edge-to-edge gluing



CW Complex

1 Vertex (e^0), 1 Disk (e^2)
Boundary of disk collapsed to vertex



2. CW complex

Since the Maggee-Puder [MP19] uses the CW-complex construction we focus on them.

A CW complex is built sequentially by gluing basic topological spaces called cells. An n-cell, denoted e^n, is topologically equivalent (homeomorphic) to the interior of an n-dimensional solid ball (D^n).

- **0-cells (e^0):** Points.
- **1-cells (e^1):** Open line segments. Their boundaries are two points.
- **2-cells (e^2):** Open flat disks (like the interior of a circle). Their boundary is a 1-dimensional circle (S^1).
- **3-cells (e^3):** Solid balls (the interior of a sphere). Their boundary is a 2-dimensional sphere (S^2).

The space formed after attaching all cells up to dimension n is called the **n-skeleton**, denoted X^n.

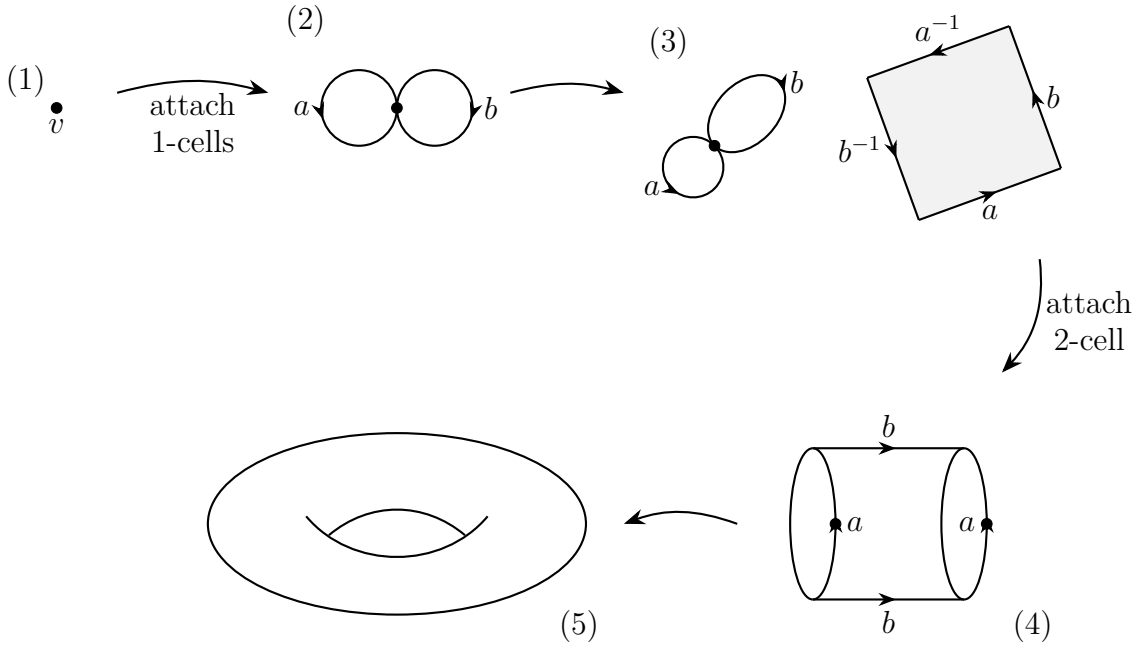
Example 2.1 (torus)**.** A torus (a donut shape) can be perfectly described using a single 0-cell, two 1-cells, and one 2-cell. We build it skeleton by skeleton:

Step 1: The 0-Skeleton (X^0) We start with a single 0-cell (a point). Let's call this vertex v. *Cell count so far: One 0-cell.*

Step 2: The 1-Skeleton (X^1) We take two 1-cells (line segments), which we will call a and b. We attach the boundaries of both a and b to the single vertex v. This bends the line segments into circles, creating a space that looks like a figure-eight or a "bouquet" of two circles ($S^1 \vee S^1$). *Cell count so far: One 0-cell, two 1-cells.*

Step 3: The 2-Skeleton (X^2) We take a single 2-cell (e^2), which is a disk. We can think of this disk as a flexible square polygon. We glue the boundary of this square to

the 1-skeleton by tracing the perimeter along the loops a and b. To get a torus, we glue the edges according to the sequence $aba^{-1}b^{-1}$ (meaning: follow a forward, b forward, a backward, b backward). *Final Cell count: One 0-cell, two 1-cells, one 2-cell.*



building a CW-complex. Let us sketch you to build a CW complex X is by the following sequential steps:

(1) The 0-skeleton (X^0): We begin by taking a discrete set of points. This set of points is the 0-skeleton, X^0. Each point is a 0-cell (e^0). That is a set of points with the discrete topology.

(2) The Inductive Assumption (X^{n-1}): Assume we have successfully constructed the $(n-1)$-skeleton, X^{n-1}. This space contains all the 0-cells, 1-cells, up to $(n-1)$-cells.

(3) Gluing maps: This is done by continuous maps from the boundary of the new n-cells into X^{n-1}. To build the next dimension, we gather a collection of closed n-disks, denoted D^n_α (where α is just an index labeling each disk). For every disk D^n_α, we must choose an attaching map:

$$\phi_\alpha : S^{n-1}_\alpha \to X^{n-1}$$

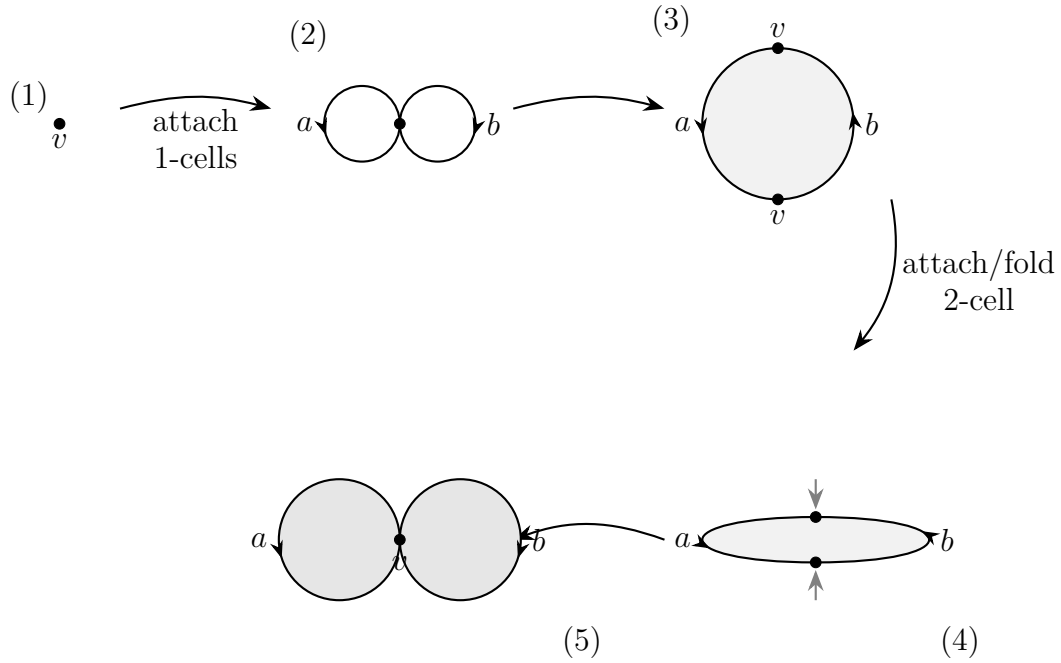
This is a continuous function that dictates exactly how the boundary of the new n-disk (S^{n-1}_α) wraps around and glues onto the existing $(n-1)$-skeleton.

(4) Gluing process: We physically attach the n-disks to X^{n-1}. Concretely, this is done using a quotient space. We take the disjoint union of X^{n-1} and all your new disks, and we identify every point x on the boundary of a disk with its target destination $\phi_\alpha(x)$ in the skeleton.

$$X^n = \frac{X^{n-1} \coprod_\alpha D^n_\alpha}{x \sim \phi(x)}$$

This new space is the n-skeleton, X^n, where $\phi : \sqcup_\alpha S^{n-1}_\alpha \to X^{n-1}$ such that restricted to S^{n-1}_α acts by ϕ_α.

Example 2.2 (surface with boundary)**.** Suppose we have the same one-skeleton as in the torus above. Instead of gluing the boundary of the 2-cell (D^2) to the 1-skeleton to form the torus, we glue the boundary to form a surface with two boundary components.



3. The Euler Characteristic (χ)

For any finite CW complex, the Euler characteristic is defined as the alternating sum of the number of cells in each dimension. If c_n represents the number of n-cells, the formula is:

$$\chi = \sum_{n\geq 0}(-1)^n c_n = c_0 - c_1 + c_2 - c_3 + \dots \tag{1}$$

Because topological spaces can be decomposed into CW complexes in many different ways, it is a profound theorem that χ is a topological invariant; it remains the same no matter how you divide the space into cells.

The 2-Dimensional Surface Case When dealing strictly with a 2-dimensional CW complex (like a surface), we only have 0-cells, 1-cells, and 2-cells. Historically, these are known as Vertices ($V = c_0$), Edges ($E = c_1$), and Faces ($F = c_2$). The formula simplifies to the familiar polyhedral formula:

$$\chi = V - E + F \tag{2}$$

For our torus constructed above: $\chi = 1 - 2 + 1 = 0$.

4. Orientable Surfaces: Genus, Components, and Boundaries

A surface is **orientable** if it has a consistent "top" and "bottom" (or inside/outside) everywhere, meaning you cannot trace a path that brings you back mirror-reversed (i.e., it contains no Möbius strips).

For orientable surfaces, we can calculate the Euler characteristic using macroscopic features rather than counting individual cells. The topology of a closed, orientable surface is entirely dictated by:

- k: The number of **connected components** (separate pieces).
- g: The total **genus** (number of holes/handles across the surface).

The Euler characteristic is then determined by:

$$\chi = 2k - 2g \tag{3}$$

Accounting for "Faces" as Boundaries: If we remove a face (a 2-cell) from a closed surface, we create a hole with an edge, known as a **boundary component**. Because removing a face directly subtracts 1 from the $V - E + F$ sum, every boundary component decreases the Euler characteristic by exactly 1.

If a surface has b boundary components, the complete formula becomes:

$$\chi = 2k - 2g - b \tag{4}$$

Thus, you can find the Euler characteristic from the bottom-up (counting cells: V, E, F) or from the top-down (counting topology: components k, genus g, boundaries b). Both methods will always yield the same invariant number.