Additional Notes on Implicit Differentiation

Oct, 2025

Motivation

Any equation in n variables (say, in $\mathbb{R}^n$) determines a hypersurface in $\mathbb{R}^n$.

We can always represent an equation in the form

$$F(x^1, \dots, x^n) = 0 \quad \text{where } F : U \subset \mathbb{R}^n \to \mathbb{R}.$$

That is, a level set of a function F.[1]

We are interested in studying the shape of the surface. We measure smoothness of the surface in terms of derivatives. But what, how, where can we find the derivative at a point on the surface?

The implicit function theorem gives us a sufficient condition for doing so. It says that if $x = (y^1, \dots, y^n) \in U$ such that F is continuous on a nbhd U with $F(y^1, \dots, y^n) = 0$ (i.e. y is a point on our surface), and

$$\frac{\partial F}{\partial x^n}(y) \neq 0,$$

then we can say something.

That is we can say that the level surface $F(x^1, \dots, x^n) = 0$ "looks like" the graph of a differentiable function around y. Precisely, there exist open sets $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^{n-1}$ and $G : U \to \mathbb{R}$ differentiable such that

$$\begin{aligned} &\{(x^1, \dots, x^n) \in V \mid F(x^1, \dots, x^n) = 0\} \\ &= \{(x^1, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})) \mid (x^1, \dots, x^{n-1}) \in U\}. \end{aligned}$$

That is, the level set looks like a graph on a small portion where it intersects V.

Moreover, the implicit function theorem tells us

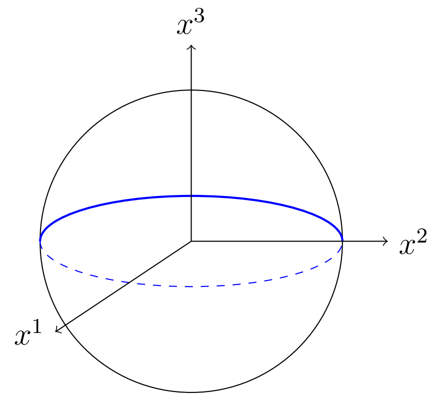
$$\frac{\partial g}{\partial x^i}(y) = -\frac{\frac{\partial F}{\partial x^i}(y)}{\frac{\partial F}{\partial x^n}(y)} \quad \text{for } i = 1, \dots, n-1.$$

[1]By $x^1, .., x^n$ we mean the coordinate functions not raised to the power. This was the notation used in a Differential Geometry course I was taking at the time of writing.

When we implicitly differentiate, we say $x^n = g(x^1, \ldots, x^{n-1})$, make this substitution in our equation $F(x^1, \ldots, x^n) = 0$, and compute partial derivatives of g.

Example 0.1. We would like to understand the shape of the unit sphere

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 0, \quad \text{at } (0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}).$$



Set

$$F(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2.$$

One checks

$$\frac{\partial F}{\partial x^3}(x^1, x^2, x^3) = 2x^3, \quad \text{and so } \frac{\partial F}{\partial x^3}(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = 2 \cdot \tfrac{1}{\sqrt{2}} \neq 0.$$

By the implicit function theorem we obtain a differentiable $g : U \subset \mathbb{R}^2 \to \mathbb{R}$.

Here $V = \{(x^1, x^2, x^3) \mid x^3 > 0\}$, upper half-sphere, and g is the upper hemisphere.

$$\frac{\partial g}{\partial x^1} = \frac{-2x^1}{2x^3} = \frac{-x^1}{x^3}, \quad \frac{\partial g}{\partial x^2} = \frac{-2x^2}{2x^3} = \frac{-x^2}{x^3}.$$

At $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ we compute:

$$\frac{\partial g}{\partial x^1}(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = 0, \qquad \frac{\partial g}{\partial x^2}(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = -1.$$

Thus,

$$\frac{\partial g}{\partial x^1}(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = 0$$

so there is no change in height for small perturbations in the x^1-direction, while

$$\frac{\partial g}{\partial x^2}(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = -1$$

so downward movement if one moves in positive x^2-direction.