Zeros of Stable NC-Polynomials

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Introduction

Classically, a polynomial in one complex variable with no zeroes in the closed unit disk $\{|z| \leq 1\}$ has no zeroes in some open disk with radius $R > 1$. This behavior carries over to polynomials in several complex variables with no zeroes in the ℓ^∞-closed-unit-ball in $\mathbb{C}^g$. The purpose of this exposition is to showcase how the same behavior carries over to nc-polynomials. In the process, we see how the behavior of nc-polynomials are determined by evaluation up to some large finite size that depends on the degree of the polynomial.

NC-Polynomials

We let $x = (x_1, \ldots, x_g)$ denote a g-tuple of non-commuting indeterminates, and $\mathbb{F}_g^+$ the free monoid on g symbols $\{1, 2, \ldots, g\}$, with neutral element denoted $\emptyset$. Recall a monoid is almost a group, but every element need not have an inverse. Given a word $w = i_1 i_2 \cdots i_k \in \mathbb{F}_g$, we write

$$x^w = x_{i_1} x_{i_2} \cdots x_{i_k}.$$

We let $\mathbb{C}\langle x_1, x_2, \ldots, x_g\rangle = \mathbb{C}\langle x\rangle$ denote the algebra of non-commuting polynomials in g indeterminates $x_1, \ldots, x_g$. As a set this is the vector space of all formal linear combinations of monomials x^w, $w \in \mathbb{F}_g^+$. And multiplication is done by concatenation of monomials in. In fact, this is a $*$-algebra where we define $*$ on a word $w = i_1 i_2 \cdots i_k$ by

$$(x^w)^* := x_{i_k}^* \cdots x_{i_2}^* x_{i_1}^*, \qquad (x_j^*)^* := x_j.$$

One checks that $*$ defines an involution on $\mathbb{C}\langle x, x^*\rangle$.

Since the set of monomials forms a basis for $\mathbb{C}\langle x, x^*\rangle$ (by construction) every nc-polynomial has a unique expression

$$p(x, x^*) = \sum_{w \in \mathbb{F}_{2g}^+} a_w x^w,$$

where $x^\emptyset := 1$.

We are also interested in polynomials with matrix coefficients; we define

$$M_k(\mathbb{C})\langle x, x^*\rangle := M_k(\mathbb{C}) \otimes \mathbb{C}\langle x, x^*\rangle,$$

the algebraic tensor product of $*$-algebras over $\mathbb{C}$. Elements of $M_k(\mathbb{C})\langle x, x^*\rangle$ can be written

$$\sum_{i=1}^{k} A_i \otimes p_i(x, x^*) = \sum_i A_i \otimes \Big(\sum_w a^i_w x^w\Big) = \sum_w \widetilde{A}_w x^w,$$

where $\otimes$ denotes the algebraic tensor product. We define

$$\deg(p) := \max\{\, |w| : A_w \neq 0\,\}.$$

Evaluation

So far we have only considered $\mathbb{C}\langle x, x^*\rangle$ and $M_k(\mathbb{C})\langle x, x^*\rangle$ as formal algebraic objects. For purposes of analysis we want to evaluate these nc-polynomials on g-tuples of matrices

$$X = (X_1, \dots, X_g) \in (M_d(\mathbb{C}))^g$$

for all $d \in \mathbb{N}$, or even for $X \in \mathcal{A}^g$ where $\mathcal{A}$ is a $*$-algebra of operators.

Informally, given $X \in M_d(\mathbb{C})^g$ and $p \in M_k(\mathbb{C})\langle x, x^*\rangle$, $p(X)$ is just replacing x_i, x_i^* with X_i, X_i^*.

Formally, given a tuples $X \in M_d(\mathbb{C})^g$ we define a $*$-homomorphism

$$\widehat{X} : \mathbb{C}\langle x, x^*\rangle \to M_d(\mathbb{C})$$

by

$$\widehat{X}(p) = \sum_w a_w X^w, \qquad p = \sum_w a_w x^w..$$

And

$$I_k \otimes \hat{X} : M_k(\mathbb{C}) \otimes \mathbb{C}\langle x, x^*\rangle$$

by

$$(I_k \otimes \hat{X})(\sum A_w \otimes p_w) = \sum A_w \otimes \hat{X}(p_w)$$

Norms

A key role in this discussion is played by operator-space norms that preserve the structure across all matrix levels. For this exposition we make use of the *minimal* operator-space structure that one can equip on $\mathbb{C}^g$, obtained by starting with the ℓ^∞-norm.

In this setting, the natural multivariate generalization of the closed unit disk for nc-polynomials is

$$\bigcup_{d=1}^{\infty} \left\{ Z \in M_d(\mathbb{C})^g : \ \|Z\|_{\min(\ell^\infty)} \leq 1 \right\},$$

where

$$\|Z\|_{\min(\ell^\infty)} = \max_{1 \leq i \leq g} \|Z_i\|,$$

and each $\|Z_i\|$ denotes the usual operator norm on $M_d(\mathbb{C})$.

Stable Polynomials

We say an nc-polynomials is *stable* with respect to $\min(\ell^\infty)$ to mean

$$\det(p(Z)) \neq 0$$

for all $d \in \mathbb{N}$, $Z \in M_d(\mathbb{C})^g$ such that $\|Z\|_{\min(\ell^\infty)} \leq 1$. We evaluate an nc-polynomial p on tuples of matrices of all sizes. What is interesting is that behavior of p is only determined by evaluation up to some finite level, where the level depends on the degree of p.

Given a word $w = i_1 i_2 \cdots i_m$, by a *tail* of w we mean any string $i_j i_{j+1} \cdots i_m$ where $1 \leq j \leq m$.

Lemma. *Let $p \in M_k(\mathbb{C})\langle x, x^* \rangle$, and $p(0) = I_k$. If $\det(p(Z)) \neq 0$ for all $Z \in \bigcup_{d=1}^{\infty} M_d(\mathbb{C})^g$ such that $\|Z\|_{\min(\ell_g^\infty)} \leq 1$, then there exists an $R > 1$ (independent of level) such that $\det(p(Z)) \neq 0$ for all $Z \in \bigcup_{d=1}^{\infty} M_d(\mathbb{C})^g$ such that $\|Z\|_{\min(\ell_g^\infty)} < R$.*

Proof. Let

$$p(x, x^*) = \sum_{|w| \leq n} A_w x^w$$

where $A_w \in M_k(\mathbb{C})$ and n denotes the degree of p.

By compactness at each level m, there exits $R_m > 1$ such that $\det(p(Z)) \neq 0$ for all $Z \in M_m(\mathbb{C})^g$ such that $\|Z\|_{\min(\ell_g^\infty)} < R_m$. Indeed for m fixed, the ball $\{Z \in M_m(\mathbb{C})^g \ : \ \|Z\|_{\min(\ell_g^\infty)} \leq 1\}$ is compact and the preimage of 0 under that function $\det(p(Z))$ is closed, with the intersection between the two empty. Therefore, the distance between the two sets in $M_m(\mathbb{C})^g$ is strictly positive, so by the triangle the conclusion follows.

Let $N := \sum_{j=0}^{n} j \cdot g^j$ and

$$R := \min\{R_1, ..., R_N\} > 1$$

where the number of levels to consider depends on the degree of p.

We claim that for all N, and $Z \in M_N(\mathbb{C})$

$$\det(p(Z)) \neq 0 \quad \text{ for all } \|Z\|_{\min(\ell^\infty)} < R.$$

By way of contradiction, suppose not. That is suppose there exists $N' > N$, and $Z \in M_{N'}(\mathbb{C})^g$ such that $\|Z\|_{\min(\ell^\infty)} < R$ but $\det(p(Z)) = 0$. Since $p(Z) : \mathbb{C}^k \otimes \mathbb{C}^{N'} \to \mathbb{C}^k \otimes \mathbb{C}^{N'}$ is not invertible, it follows that $p(Z)$ has a non-trivial kernel. Let $v \in \mathbb{C}^k \otimes \mathbb{C}^{N'}$ non-zero such that

$$\sum_{|w| \leq n} (A_w \otimes Z^w) v = 0.$$

Define

$$\mathcal{K} := \operatorname{span}\Big\{ (I_k \otimes Z^w) v \ : \ w \text{ is a tail of } w_0 \text{ and } A_{w_0} \neq 0 \Big\}$$

Since $p(0) = I_k$ we have $(I_k \otimes Z^{\emptyset})v = v \in \mathcal{K}$. Also, for each $A_w \neq 0$ the number of linearly independent vectors contributed to $\mathcal{K}$ is at most $|w|$. Indeed if $w = i_1 i_2 \cdots i_{|w|}$ then we add $Z_{i_{|w|}}v$, $Z_{i_{|w|-1}}Z_{i_{|w|}}v, \cdots, Z_{i_1} \cdots Z_{i_{|w|-1}}Z_{i_{|w|}}v$ to the set before taking the span. It follows that $N = \sum_{j=0}^{n} j \cdot g^j$ is an upper bound for the dimension of $\mathcal{K}$.

Denote the orthogonal projection onto $\mathcal{K}$ by $P_{\mathcal{K}}$. Let us construct the tuple $T = (T_1, \cdots, T_g)$ from $T_i : \mathcal{K} \to \mathcal{K}$ by

$$T_i(u) := P_{\mathcal{K}}(I_k \otimes Z_i)|_{\mathcal{K}}(u).$$

First, observe that

$$p(T)v = p(Z)v = 0.$$

Indeed, it suffices to show that

$$T^w v = (I_k \otimes Z^w)v \tag{1}$$

for all w such that $A_w \neq 0$. Becuase then $(A_w \otimes I_{N'})T^w v = (A_w \otimes I_{N'})(I_k \otimes Z^w)v = (A_w \otimes Z^w)v$.

We proceed by induction on the length of monomial words. If $|w| = 1$, then $w = i$ and

$$T_i(v) = P_{\mathcal{K}}(I_k \otimes Z_i)|_{\mathcal{K}}(v) = P_{\mathcal{K}}(I_k \otimes Z_i)(v) = (I_k \otimes Z_i)(v).$$

The first equality follows from $v \in \mathcal{K}$, and the second from $Z_i v \in \mathcal{K}$. For the inductive step, suppose equation 1 holds for all words w of length $m \geq 1$ such that $A_w \neq 0$, and let $|w| = m + 1$. That is $w = i_1 \cdots i_{m+1}$. Then

$$\begin{aligned} T_w(v) &= T_{i_1} \cdots T_{i_m} T_{i_{m+1}}(v) \\ &= T_{i_1}(I_k \otimes Z_{i_2} \cdots Z_{i_{m+1}})(v) \\ &= (I_k \otimes Z^w)(v) \end{aligned}$$

where the second equality follows from the inductive step, and the last from the construction of $\mathcal{K}$.

But now $T = (T_1, \cdots, T_g)$ is a g-tuple acting on a space with dimension at most N, such that $\det(p(T)) = 0$ and

$$\|T\|_{\min(\ell^\infty)} = \max_{i=1}^{g} \|T_i\| = \max_{i=1}^{g} \|P_{\mathcal{K}}(I_k \otimes Z_i)|_{\mathcal{K}}\| \leq \max_{i=1}^{g} \|I_k \otimes Z_i\| = \max_{i=1}^{g} \|Z_i\| < R$$

contradicting our choice of R.

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